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Concrete Quantum Logics†

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Concrete quantum logics are quantum logics which allow for a set representation. They seem to be of significant conceptual value within quantum axiomatics and they play an important role in the theory of orthomodular structures as set-representable orthomodular posets or lattices and they also sometimes constitute a "domain" for investigations in "noncommutative" measure theory. This paper presents a survey of recent results on this class of logics. Stress is put on the algebraic and measure-theoretic aspects. Several open questions relevant to the logicoalgebraic foundation of quantum theories are posed.

INTRODUCTION

A quantum logic—a "logic" of the events of a quantum experiment—is often assumed to be an orthomodular poset [14, 51, 63]. In contrast to "classical" logics (Boolean algebras), quantum logics do not have to allow for a set representation. *Those quantum logics which allow for a set representation are called concrete*. The formal definition reads as follows.

1. Definition. A concrete (quantum) logic is a couple (Ω, Δ), where Ω is a nonvoid set and $\Delta \subset \exp \Omega$ is a collection of subsets of Ω which is partially ordered by inclusion and which is subject to the following requirements:

(i) $\emptyset \in \Delta$.
(ii) $A \in \Delta \Rightarrow \Omega \backslash A \in \Delta$ (here $\backslash$ stands for the set complement in Ω).
(iii) $A \in \Delta$, $B \in \Delta$, $A \cap B = \emptyset \Rightarrow A \cup B = \in \Delta$.

A concrete logic is represented by the collection Δ. However, very often

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we have to refer to the underlying set Ω, too. Thus, we usually consider Δ together with Ω and refer to the pair (Ω, Δ) as concrete logic. The following simple but useful characterization of concrete logics is due to Gudder [14]. The proof could be done by the standard Boolean technique. [Recall that by *a state* on a (general) quantum logic we mean a finitely additive probability measure on it. Let us denote by $\mathcal{S}(L)$ the set of all states on L, and by $\mathcal{S}_2(L)$ the set of all two-valued states on L.]

Proposition 1. Let L be a quantum logic. Then L is (isomorphic to) a concrete logic if, and only if, L possesses an order determining set of two-valued states [i.e., for each $a, b \in L$, $a \leq b$, there exists a state $s \in \mathcal{S}_2(L)$ such that $s(a) = 1$ and $a(b) = 0$].

1. WHICH LOGICS *ARE NOT* CONCRETE?

Obviously, the lattice of projections in a Hilbert space $L(H)$ is not concrete whenever dim $H \geq 3$. Indeed, $L(H)$ does not possess any two-valued state (the fact that $\mathcal{S}_2(L(H)) = \emptyset$ follows easily from the Gleason's theorem [10]; see also ref. 7 for a detailed analysis). There are even finite logics which are not concrete. Some of them can be constructed as sublogics of $L(H)$ [25, 59, 60]. Other examples can be constructed by the Greechie pasting technique [12]. Figure 1 presents a typical one [26]. One can easily find elements a, b of this logic such that the equalities $s(a) = 1$, $s(b) = 0$ cannot be simultaneously satisfied. Thus, this logic is not concrete.

2. WHICH LOGICS *ARE* CONCRETE?

The class of concrete logics is quite large. Let us demonstrate this by exhibiting various examples. Some of them will be revisited later for a more

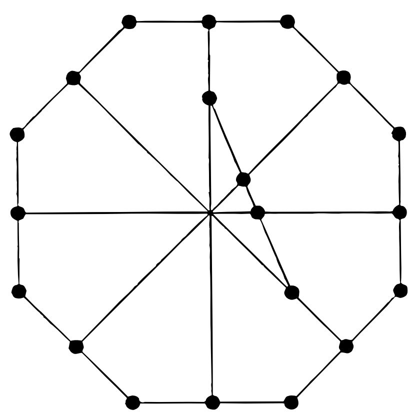

Fig. 1.

detailed study. In accord with the general definition of state, a state on a concrete logic (Ω, Δ) is a mapping s: $\Delta \to \langle 0, 1\rangle$ such that $s(\Omega) = 1$ and $s(A \cup B) = s(A) + s(B)$, provided $A, B \in \Delta$ and $A \cap B = \emptyset$.

(1) *Boolean algebras*. A Boolean logic (i.e., a Boolean algebra) is a concrete logic (the Stone representation makes each Boolean algebra a concrete logic). The states on Boolean algebras are exactly the finitely additive probability measures.

(2) *Subsets of an even cardinality—the logic* Ω_{even}. Let $\Omega = \{1, 2, \ldots, 2l - 1, 2l\}$ be a set of an even cardinality. Then (Ω, Δ), where $\Delta = \{A | \text{card } A$ is an even number$\}$, is a concrete logic [14]. Let us denote it by Ω_{even}. Observe that

(i) Ω_{even} is a lattice if and only if card $\Omega \leq 4$.

(ii) If card $\Omega \geq 6$, then all two-valued states are "Dirac measures" (i.e., measures sitting at points [e.g., [44, 58]].

(3) *Partitions into large sets together with singletons*. Let $\Omega = \{(x, y) \in R^2 | x^2 + y^2 \leq 1\}$, and let $A = \{(x, y) \in \Omega | x \geq 0\}$, $B = \{(x, y) \in \Omega | y \geq 0\}$. Then (Ω, Δ), where $\Delta = \{\emptyset, \Omega, A, A', B, B'$, finite subsets of Ω, cofinite subsets of $\Omega\}$, is a concrete logic.

(4) *Density logic*. Let $N = \{1, 2, 3, \ldots\}$ be the set of natural numbers. Let $\Omega = N$ and let $\Delta \subset \exp \Omega$ consists of those sets which have a density, i.e., $A \in \Delta$ if $\lim_{n\to\infty} [\text{card}(A \cap \{1, 2, \ldots, n\})/n]$ exists. The function d: $\Delta \to \langle 0,1\rangle$ defined by setting $\lim_{n\to\infty} [\text{card}(A \cap \{1, 2, \ldots, n\})/n] = d(A)$ becomes a state on Δ. (This construction can be generalized to measure spaces with a σ-finite measure. The density logic has interesting state extension properties, see Section 5.4).

(5) *Rational area logics*. Let $\Omega = \langle 0, 1\rangle^2$, and let μ be the Lebesgue measure on Ω. Let

$$\Delta = \langle A | A \subset \langle 0, 1\rangle^2, A \text{ is } \mu\text{-measurable}, \mu(A) \text{ is a rational number}\}$$

Then (Ω, Δ) is a concrete logic. As regards the states on (Ω, Δ), observe the following fact, which is quite typical for concrete logics and which is not shared by Boolean algebras: There are states on (Ω, Δ) which are *not* subadditive. [Recall that a state $s \in \mathcal{S}(\Delta)$ is called *subadditive* if for each $A, B \in \Delta$ there is a $C \in \Delta$ such that $C \supset A \cup B$ and, moreover, $s(C) \leq s(A) + s(B)$.] In fact, we can easily see that the following statement is true. Let us fix a set $D \in \Delta$ with $\mu(D) > 0$, and define the set function s:$\Delta \to \langle 0, 1\rangle$ by setting $s(A) = \mu(A \cap D)/\mu(D)$. Then s is subadditive if and only if $\mu(D)$ is a rational number.

(6) *Infimum faithful logics*. Let L be a (general) logic. Let L fulfil the following condition [24]. Suppose $a, b \in L$. Then $a \leftrightarrow b$ (a is compatible with b in L [e.g., 63] if and only if $a \wedge b$ exists in L. It can be proved that

L is concrete [36]. Note that the previous examples (3)–(5) are infimum faithful logics.

(7) *The Kalmbach concrete logics*. It can be shown that each lattice can be lattice-theoretically embedded into a concrete lattice logic. This fact was first observed by Kalmbach [21] and proved in detail by Harding [18] and Navara [29].

(8) *The free lattice logic with two generators is concrete*. The free lattice logic is a synonym for a free orthomodular lattice. The free orthomodular lattice with two generators is concrete ([29]). This rather interesting fact does not seem to be explicitly contained in monographs on algebraic theory of orthomodular lattices [3, 22]. It is not known if the free orthomodular lattice with three (and more) generators is concrete.

(9) *Concrete lattice logics form a variety of algebras*. Concrete lattice logics form a variety when viewed as a class of orthomodular lattices [11]. This variety is not finitely based, but allows for a relatively transparent equational description [26].

(10) *Concrete logics and a group representation*. Each group is an automorphism group of a concrete (lattice) logic [37]. It is not known how far one could go in generalizing this result, but, of course, the class of Boolean algebras is too restrictive [13].

(11) *Permanence properties of concrete logics*. Concrete logics are closed under the formation of sublogics, products and ultraproducts. They also allow for a tensor product with Boolean algebras in the category of concrete logics [8, 9, 55]. In particular, each concrete logic can be enlarged to a concrete logic with a given center.

3. COMPATIBILITY (RESP. NONCOMPATIBILITY) RELATION IN CONCRETE LOGICS

One of the conceptual advantages of concrete logics is that the physically important compatibility relation can be transparently expressed (see Proposition 3.1 below). On the other hand, concrete logics may be intrinsically as far from compatibility regular logics as general logics (for the definition of regularity, see ref. 51, Def. 1.3.26).

Proposition 3.1 [51]. The collection $A_1, A_2, \ldots, A_n \subset \Delta$ in a concrete logic (Ω, Δ) is compatible if and only if for each n-tuple $d = (d_1, d_2, \ldots, d_n) \in \{-1, 1\}^n$ we have $\bigcap_{i \leq n} A_i^{d_i} \in \Delta$ (as usual, we understand $A^1 = A$ and $A^{-1} = A'$). In particular, $A \leftrightarrow B$ in $\Delta \Leftrightarrow A \cap B \in \Delta \Leftrightarrow A \cup B \in \Delta$ (thus, (Ω, Δ) is Boolean if and only if $A \cap B \in \Delta$ [resp. $(A \cup B \in \Delta)$] for each pair $A, B \in \Delta$).

Proposition 3.2 [51]. Let n be a natural number, $n \geq 2$. Let card $\Omega = 2^n$. Then the concrete logic Ω_{even} has the following property: There exists a noncompatible collection $\{A_1, A_2, \ldots, A_n) \subset \Omega_{\text{even}}$ such that each of its proper subcollections is compatible. Thus, Ω_{2^n} is not regular for any $n \in N$.

4. CLASSES OF CONCRETE LOGICS WITH PECULIAR INTRINSIC AND MEASURE-THEORETIC PROPERTIES

This section offers further information on the richness of the class of concrete logics. Let us consider the following classes:

- B the class of Boolean algebras
- B_1 the class of all concrete logics which have all states subadditive
- B_2 the class of all "compact-like" concrete logics: A logic (Ω, Δ) is called compact-like if for any pair $A, B \in \Delta$ there is a finite collection $\{C_1, C_2, \ldots, C_n\} \subset \Delta$ such that $A \cap B = \cup_{i \leq n} C_i$
- B_3 the class of all infimum faithful logics
- B_4 the class of all concrete logics

Theorem 4.1 [36, 45]. We have the inclusions

$$B \subseteq B_1 \subseteq B_2 \subseteq B_3 \subseteq B_4$$

and all these inclusions are proper.

The most interesting assertion of the above theorem seems to be the result that the inclusion $B \subseteq B_1$ is proper. This may reach beyond the theory of quantum logics. The result was established in ref. 45 with substantial help from the techniques developed in refs. 27 and 34. The following questions remain open.

(1) Can every logic be embedded, in a compatibility-preserving manner, into a logic of the class B_1? (This question may have bearing on the quantum axiomatic. Since, as one checks easily, a subadditive state is necessarily Jauch–Piron (see also ref. 45), a positive answer to this question would often allow us to restrict to concrete Jauch–Piron logics; Observe that the σ-additive version of this problem has been answered in the positive [6]).

(2) Let us call (Ω, Δ) σ-complete if Δ is closed under the formation of unions of countable pairwisely disjoint sets. If (Ω, Δ) is σ-complete and if it belongs to B_1, does (Ω, Δ) have to be Boolean? (If (Ω,Δ) is $2^{\aleph_0}$ complete (= continuum-complete), then the answer is yes [45].)

5. MORE ON CONCRETE LOGICS—ARE THEY "ALMOST" BOOLEAN?

In this section we indicate some other lines of research on concrete logics. The results obtained shed light on the relations of concrete logics to general logics and Boolean algebras.

5.1. Constructions of Compact-Like Concrete Logics with Uniformly Bounded Covering Type

Theorem [28]. Given a natural number $n \in N$, there exists a concrete logic (Ω, Δ) such that, for each couple $A, B \in \Delta$, $A \cap B = \cup_{i \leq n} C_i$ for some $\{C_1, C_2, \ldots, C_n\} \subset \Delta$, and, moreover, there is a couple $D, E \in \Delta$ so that $D \cap E$ cannot be written as a union of strictly fewer than n sets of the logic Δ.

This result introduces a kind of covering dimension into the realm of concrete logics. One deals with a new type of orthomodular combinatorics. All constructions involved are infinite and nonlattice.

5.2. General Logics as Epimorphic Images of Concrete Ones

We have the following result (a version of the Loomis–Sikorski theorem for logics).

Theorem [54]. Each logic is an epimorphic image of a compact-like concrete logic, and, moreover, the epimorphism may be required such that it preserves the compatibility relation in a weak sense (a logic morphism l: $L_1 \to L_2$ preserves the compatibility relation in a weak sense if for each compatible set $\{b_1, b_2, \ldots, b_n\} \subset L_2$ there is a compatible set $\{a_1, a_2, \ldots, a_n\} \subset L_1$ such that $l(a_i) = b_i$). On the other hand, if L_1 is concrete and l: $L_1 \to L_2$ is a logic epimorphism which preserves compatibility in the stronger sense (i.e., if all preimages of finite compatible sets are compatible), then L_2 is concrete, too.

It does not seem to be known if the latter theorem remains true if we require the weak preservation of compatibility for all (not only finite) sets.

5.3. Generalized Stone Representations.

A nonconcrete logic cannot be embedded into a concrete one, of course. In the attempt to construct at least weaker kinds of embeddings, generalized Stone representations have been investigated [19, 42, 49, 62, 65]. The most complete treatment seems to be ref. 62, where one also finds some relevant open problems.

5.4. Extensions of States on Concrete Logics

Suppose that a concrete logic L is a sublogic of a logic K. Suppose that $s \in \mathcal{S}(L)$. Then if L is Boolean, s can be extended over K ([PtExt]). If L is not Boolean, the extension may not exist. It is slightly surprising, however, that in some relatively complex non-Boolean cases we can find extensions. Here is an example (an unpublished result of the author).

Theorem. Let H be a separable Hilbert space and let $B = \{b_i | i \in N\}$ be an orthonormal basis of H. Let Δ be the density concrete logic on B. Let d be the density state on Δ (see Example 4 in Section 2). Then d can be extended over $L(H)$.

5.5. How to Characterize the State Spaces of Concrete Logics?

A characterization of the state space (resp., a characterization of the two-valued-state space) of concrete logics does not seem to be known. It is easily seen, however, that not all compact convex subsets in $\langle 0, 1\rangle^P$, where P is a set, may serve as state spaces of concrete logics. Also, it should be observed that not all pure states on concrete logics have to be two-valued [35]. This may occur even for finite concrete lattice logics [32].

5.6. Gudder's Integral

An interesting study with concrete logics has been the research on generalized integration [15]. Even for finite logics some original nontrivial problems have appeared. Most of the questions have been significantly contributed to or solved in refs. 15, 33, 35 and 61. The results culminated in a remarkable paper by Gudder and Zerbe [16]. The latter paper inspired a vast generalization [30], which, however, still did not reach a complete characterization for the additivity of Gudder's integral to hold [30, 31].

6. CONCRETE σ-COMPLETE LOGICS

The concrete σ-complete logics have been investigated in measure theory ever since [e.g., 1, 17, 38]. Their alternative names have been σ-classes, Dynkin systems, etc. They proved to be instrumental in analysis and probability. In this exposition we would like to introduce an interesting topical investigation in concrete σ-complete logics. It concerns the σ-complete logics generated by balls in metric spaces. The results could be applicable in other areas of mathematics, and possibly within the foundation of quantum mechanics.

Definition 6.1. A concrete logic (Ω, Δ) is called σ-complete if it is closed under the formation of unions of countable pairwise disjoint families of sets from Δ. By a *state* on the σ-complete concrete logic (Ω, Δ) we then mean a countably additive probability measure on Δ.

Obviously, given a subset S of $\exp \Omega$, there is a smallest concrete σ-complete logic $(\Omega, \Delta_\sigma(S))$ which contains S. Let us call it the logic generated by S. This logic may or may not coincide with the Boolean σ-algebra (on Ω) generated by S. When it does, we easily see that if two probability measures μ_1, μ_2 agree on S, they must agree on the entire $\Sigma(S)$. Indeed, it is easy to check that if two states on $\Delta_\sigma(S)$ agree on S, they must be identical. Let us ask when $\Delta_\sigma(S) = \Sigma(S)$. Here are a few results on how the situation looks in R^n when we make a natural choice of S.

Suppose that S is a basis for open sets in R^n. Then $\Sigma(S)$ coincides with all Borel sets. If S is multiplicative (i.e., if S is closed under the formation of intersections), then it can be shown that $\Delta_\sigma(S) = \Sigma(S)$ [e.g., 39]. Thus, if S consists of open rectangles, which seems to be the most natural choice for S, the situation is clear. A less clear situation occurs when we make the second most natural choice for S—the set of all open balls. Then we face an interesting and quite nontrivial problem. This problem was partially solved by Olejček [40, 41] and then fully resolved independently by Zelený [64] and Jackson and Mauldin [20]. It should be noted that if we formulate the same question in a general metric space taken instead of R^n, the answer can be in the negative even for a compact 0-dimensional space [5].

Theorem 6.2 [20, 41, 64]. Let S be the collection of all open balls in R^n. Then $\Delta_\sigma(S) = \Sigma(S)$. Thus, the logic $\Delta_\sigma(S)$ coincides with the σ-algebra of Borel sets. A corollary: If two probability Borel measures agree on all open balls in R^n, they have to be identical.

It should be noted that the corollary remains true even if we do not assume the probability Borel measures to agree on all balls, but only on a suitable subcollection. Also, the corollary may remain true even in spaces where $\Delta_\sigma(S) \neq \Sigma(S)$. It follows from the following results [see ref. 53; the technique therein allows for generalizations to a separable Hilbert space; observe that $\Delta_\sigma(S) \neq \Sigma(S)$ in this case].

Theorem 6.3 [53]. Let H be a separable Hilbert space and let $p \in H$. Let S be the collection of all open balls which have the point p in their boundary. Then, if two probability Borel measures on $B(H)$ agree on S, they have to be identical.

It does not seem to be known if Theorem 6.3 can be generalized to the case of all Banach spaces. It should be observed, however, that the corollary formulated in Theorem 6.2 remains true even for all Banach spaces. This remarkable result was recently proved by Preiss and Tišer [43]. Let us formulate it in the conclusion of this survey. It should be observed that the attempt to obtain this result via $\Delta_\sigma(S) = \Sigma(S)$ would be in vain—in ref. 23 the authors show that $\Delta_\sigma(S) \neq \Sigma(S)$(= Borel set) even for a Hilbert space (see also ref. 56 for other results in this line).

Theorem 6.4 [43]. Let B be a separable Banach space and let two probability measures μ_1, μ_2 on the Borel algebra of B agree on the collection of all open balls in B. Then $\mu_1 = \mu_2$.

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REFERENCES

[1] Bauer, H., *Probability Theory and Elements of Measure Theory*, Academia Press, 1981.
[2] Beltrametti, E., and Cassinelli, G., *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts, 1981.
[3] Beran, L., *Orthomodular Lattices. Algebraic Approach*, Academia, Prague and Reidel, Dordrecht, 1984.
[4] Bruns, G., Greechie, R. J., Harding, J., and Roddy, M., Completions of orthomodular lattices, *Order* **7** (1990), 67–76.
[5] Davies, R. O., Measures not approximable or not specifiable by means of balls, *Mathematika* **18** (1971), 157–160.
[6] DeLucia, P., and Pták, P., Quantum probability spaces that are nearly classical, *Bull. Polish Acad. Sci. Math.* **40** (1992), 163–173.
[7] Dvurečenskij, A., *Gleason's Theorem and Application*, Kluwer, Dordrecht, 1993.
[8] Dvurečenskij, A., On the tensor product of orthoalgebras, *Trans. Am. Math. Soc.* **347** (1995), 1043–1057.
[9] Foulis, D., and Pták, P., On the tensor product of Boolean algebras and orthoalgebras, *Czechoslovak. Math. J.* **45** (1995), 117–126.
[10] Gleason, A., Measures on the closed subspace of a Hilbert space, *J. Math. Mech.* **6** (1957), 885–894.
[11] Godowski, R., Varieties of orthomodular lattices with a strongly full set of states, *Demonstratio Math.* **14** (1981), 725–733.
[12] Greechie, R., Orthomodular lattices admitting no states, *J. Comb. Theory* **10** (1971), 119–132.
[13] Groot, J. de, and McDowell, R. H., Autohomeomorphism groups of 0-dimensional spaces, *Compositio Math.* **15** (1963), 203–209.
[14] Gudder, S., *Stochastic Methods of Quantum Mechanics*, North-Holland, Amsterdam, 1979.
[15] Gudder, S., Quantum probability spaces, *Proc. Am. Math. Soc.* **21** (1969), 296–302.
[16] Gudder, S., and Zerbe, J., Additivity of integrals on generalized measure spaces, *J. Comb. Theory (A)* **30** (1985), 42–51.
[17] Halmos, P., *Measure Theory*, Van Nostrand, New York, 1950.
[18] Harding, J., Orthomodular lattices whose Mac Neille completions are not orthomodular, *Order* **8** (1991), 93–103.

[19] Iturrioz, L., A representation theory for orthomodular lattices by means of closure spaces, *Acta Math. Hung.* **47** (1986), 145–151.

[20] Jackson, S., and Mauldin, R. D., On the σ-class generated by open balls, to appear.

[21] Kalmbach, G., Orthomodular lattices do not satisfy any special lattice equation, *Arch. Math.* **27** (1977), 7–8.

[22] Kalmbach, G., *Orthomodular Lattices*, Academic Press, New York, 1983.

[23] Keleti, T., and Preiss D., The balls do not generate all Borel sets using complements and countable disjoint unions, to appear.

[24] Klukowski, J., On representations of Boolean orthomodular partially ordered sets, *Demonstratio Math.* **8** (1975), 405–423.

[25] Kochen, S., and Specker, E. P., The problem of hidden variables in quantum mechanics, *J. Math. Mech.* **17** (1967), 59–87.

[26] Mayet, R., Equational bases for some varieties of orthomodular lattices related to states, *Algebra Universalis* **23** (1986), 167–195.

[27] Müller, V., Jauch–Piron states on concrete quantum logics, *Int. J. Theor. Phys.* **32** (1993), 433–442.

[28] Müller, V., Pták, P., and Tkadlec, J., Combinatorial properties of concrete quantum logics, *Int. J. Theor. Phys.* **31** (1992), 843–854.

[29] Navara, M., On generating finite orthomodular sublattices, *Tatra Mountains Math. Publ.* **10** (1997), 109–117.

[30] Navara, M., When is the integral on quantum probability space additive? *Real Analysis Exchange* **14** (1989), 228–234.

[31] Navara, M., Integration on generalized measure spaces, *Acta Univ. Carolinae Math. Phys.* **30** (1989), 121–124.

[32] Navara, M., There are concrete OML's which have non-two-valued pure states, preprint, Prague (1998).

[33] Navara, M., The integral on σ-classes is monotonic, *Czechoslovak Math. J.* **20** (1984), 417–421.

[34] Navara, M., Quantum logics representable as kernels of measures, *Czechoslovak Math. J.* **46** (1986), 587–597.

[35] Navara, M., and Pták, P., Two-valued measures on σ-classes, *C̄as. pěst. mat.* **108** (1983), 225–229.

[36] Navara, M., and Pták, P., Almost Boolean orthomodular posets, *J. Pure Appl. Alg.* **60** (1989), 105–111.

[37] Navara, M., and Tkadlec J., Automorphisms of concrete logics, *Comment. Math. Univ. Carolinae* **32** (1991), 15–25.

[38] Neveu, J., *Bases Mathématiques du Calcul des Probabilités*, Masson, Paris, 1964.

[39] Neubrun, T., A note on quantum probability spaces, *Proc. Am. Math. Soc.* **25** (1970), 672–675.

[40] Olejček, V., q-sigma-algebras generated by balls, *Int. J. Theor. Phys.* **34**, (1995), 1643–1646.

[41] Olejček, V., The σ-class generated by balls contains all Borel sets, *Proc. Am. Math. Soc.* **123** (1995), 3665–3675.

[42] Ovtchinikov, P., Exact topological analogs to orthoposets, *Proc. Am. Math. Soc.* **125** (1997), 2839–2861.

[43] Preiss, D., and Tišer, J., Measures in Banach spaces are determined by their values on balls, *Mathematika* **38** (1991), 391–397.

[44] Pták, P., Hidden variables on concrete logics (extensions), *Comment. Math. Univ. Carolinae* **28** (1987), 157–163.

[45] Pták, P., Some nearly Boolean orthomodular posets, *Proc. Am. Math. Soc.* **126** (1998), 2039–2046.
[46] Pták, P., Logics with given centres and state spaces, *Proc. Am. Math. Soc.* **88** (1983), 106–109.
[47] Pták, P., Summing of Boolean algebras and logics, *Demonstratio Math.* **19** (1986), 349–357.
[48] Pták, P., Jauch–Piron property (everywhere!) in the logico-algebraic foundation of quantum theories, *Int. J. Theor. Phys.* **32** (1993), 1985–1990.
[49] Pták, P., Weak dispersion-free states and the hidden variables hypothesis, *J. Math. Phys.* **24** (1983), 839–840.
[50] Pták, P., Extensions of states on logics, *Bull. Acad. Polon. Sci. Ser. Math.* **33** (1985), 493–497.
[51] Pták, P., and Pulmannová, S., *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, 1991.
[52] Pták, P., and Pulmannová, S., A measure theoretic characterization of Boolean algebras among orthomodular lattices, *Comment. Math. Univ. Carolinae* **35** (1994), 205–208.
[53] Pták, P., and Tkadlec, J., A note on determinacy of measures, *Čas. pěst. mat.* **113** (1988), 435–436.
[54] Pták, P., and Wright, J. D. M., On the concreteness of quantum logics, *Aplikace Mat.* **30** (1985), 274–285.
[55] Riečanová, Z., Atomic orthoposets with orthomodular Macneille completions, *Tatra Mountains Math. Publ.* **2** (1993), 203–207.
[56] Riss, E., On the scooping property of measures by means of disjoint balls, *Acta Univ. Carolinae Math. Phys.* **37** (1996), 71–82.
[57] Sikorski, R., *Boolean Algebras*, Springer-Verlag, Berlin, 1969.
[58] Sultanbekov, F., Operation associated to partitions and best extension of signed measures on set logics, *Int. J. Theor. Phys.* **34** (1995), 1735–1739.
[59] Svozil, K., and Tkadlec, J., Greechie diagrams, nonexistence of measures in quantum logics and Kochen–Specker type constructions, *J. Math. Phys.* **37** (1996), 5380–5401.
[60] Svozil, K., *Quantum Logics*, Springer-Verlag, 1998.
[61] Šipoš, J., The integral on quantum probability space is monotonic, *Rep. Math. Phys.* **21** (1985), 65–68.
[62] Tkadlec, J., Partially additive states on orthomodular posets, *Coll. Math.* **LXII** (1991), 7–14.
[63] Varadarajan, V., *Geometry of Quantum Theory I*, Van Nostrand, Princeton, New Jersey (1968).
[64] Zelený, M., The Dynkin system generated by balls in R^d contains all Borel sets, to appear.
[65] Zierler, N., and Schlesinger, M., Boolean embeddings of orthomodular sets and quantum logics, *Duke Math. J.* **33** (1965), 251–262.